

BCJ Relation of Color Scalar Theory and KLT Relation of Gauge Theory

Yi-Jian Du^a, Bo Feng^{abc}, Chih-Hao Fu^b

^a*Zhejiang Institute of Modern Physics, Zhejiang University, Hangzhou, 310027, P. R. China*

^b*Center of Mathematical Science, Zhejiang University, Hangzhou, China*

^c*Kavli Institute for Theoretical Physics China, CAS, Beijing 100190, China*

E-mails: yjdu@zju.edu.cn, b.feng@cms.zju.edu.cn, zhihao.fu@cms.zju.edu.cn

ABSTRACT: We present a field theoretical proof of the conjectured KLT relation which states that the full tree-level scattering amplitude of gluons can be written as a product of color-ordered amplitude of gluons and color-ordered amplitude of scalars with only cubic vertex. To give a proof we establish the KK relation and BCJ relation of color-ordered scalar amplitude using BCFW recursion relation with nonzero boundary contributions. As a byproduct, an off-shell version of fundamental BCJ relation is proved, which plays an important role in our work.

KEYWORDS: Gauge symmetry, QCD, Supersymmetric gauge theory .

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1. Introduction

Despite the apparently straightforward algorithm laid down by perturbation field theory, computations of scattering amplitude in Yang-Mills theory has been known to be a formidable task even at tree level due to the enormous amount of Feynman graphs consist in the amplitude. In [1] a remarkable breakthrough was made by Witten from string theory inspired observations, which has lead to series of development ever since. For a review on these progress see, for example, [2]. In particular, an on-shell recursion relation devised to cope with tree amplitudes was derived by Britto, Cachazo, Feng and Witten (BCFW) [3, 4] based on analytic properties of the S-matrix. Applications of BCFW recursion technique have recently been extended to loop-levels by various groups (see for example [5]). A generalization which will be useful for our paper is the BCFW on-shell recursion relation with nonzero boundary contributions discussed in [6].

Incidentally, a perhaps even greater challenge was found in the attempt of computing gravity amplitudes. The lagrangian of Einstein gravity produces infinitely many vertices under perturbation, making a direct computation exhausting all Feynman graphs practically impossible. Parallel to the development in Yang-Mills theory, an unexpected shortcut was found from the string dual to the theory. By taking field theory limit a set of relations was found by Kawai, Lewellen and Tye (KLT) which expresses gravity amplitudes as “squares” of the seemingly unrelated color-ordered Yang-Mills amplitudes [7]. While in string theory the proof of KLT follows from monodromy [7, 8], the proof of KLT relations can as well be done from a purely field theoretical viewpoint using BCFW recursion relation [9, 10, 11, 12]. In the proof, another set of relations discovered by Bern, Carrasco and Johansson (BCJ) [13] played an essential role. As a byproduct of the proof of KLT relation, a new $(n - 2)!$ symmetric KLT relation was founded in [9] and will be used in this paper.

The BCJ relation displays merely one of the many interesting properties of color-ordered amplitudes in gauge theory. BCJ relation and other properties such as color-order reversed relation, $U(1)$ -decoupling relation and Kleiss-Kuijf (KK) relations [14] have been investigated from the point of view of string theory in [15, 16] (see also some related result in [37] from the pure spinor string theory) as well as from field theory in [17, 18, 19, 20, 21, 38]. The dual form of BCJ relation to loop level has also been discussed in [22, 23, 36].

Although the KK, BCJ and KLT relations were initially constructed to describe properties of gauge theory and gravity (pure and supersymmetric), these relations were found to function well beyond their original designs. For example, KK relations were proven to hold for gauge theory coupled to gravitons [24] at leading order and BCJ relations were shown to hold for tree amplitudes with gauge field coupled to two fermions or two scalars[25]. A more nontrivial example is the conjectured made by Bern, Freitas and Wong (BFW) [26]¹ that the full tree-level amplitude of gauge theory can be expressed in the form of a KLT relation as products of two amplitudes, where one of the amplitudes is the color-ordered gauge amplitude and the other one is the amplitude of a scalar theory that contains only cubic vertex described by a totally anti-symmetric coupling constant f^{abc} .

The conjecture of BFW is simple but not so easily derived. An initial reaction may be to use the supersymmetric KLT relation

$$\mathcal{M}^{\mathcal{N}=8} = \sum_{\alpha, \beta} A^{\mathcal{N}=4}(\alpha) \mathcal{S}[\alpha|\beta] \tilde{A}^{\mathcal{N}=4}(\beta), \quad (1.1)$$

since on one side of the equation, the full supermultiplet of $\mathcal{N} = 8$ SUGRA theory labeled by eight Grassmann variables $\eta^i, i = 1, \dots, 8$ contains both graviton and gluon, while on the other side of the equation the full supermultiplet of $\mathcal{N} = 4$ SYM theory labeled by four Grassmann variables ($\eta^i, i = 1, \dots, 4$ for A and $\tilde{\eta}^i, i = 5, \dots, 8$ for \tilde{A}) contains both gluon and scalar. Specifically, we obtain expressions of amplitudes of the desired particle types by reading off the corresponding Grassmann variable $\eta^{i_1} \dots \eta^{i_r}$

¹Recently, KK and BCJ like relations for the remaining rational function of one-loop gluon amplitudes have also been investigated in [27].

expansion coefficients with helicity of the particle prescribed by $h = \mathcal{N}/4 - r/2$. For the purpose of discussion here, let us associate $\eta^1\eta^2$ for gluons with positive helicity and $\prod_{j=3}^8 \eta^j$ for gluons with negative helicity on the left hand side of (1.1). To keep the equation consistent, on the right hand side of (1.1) a contribution of $\eta^1\eta^2$ or $\eta^3\eta^4$ must come from amplitude $A^{\mathcal{N}=4}$, whereas a contribution of zero power of η or of $\eta^5\eta^6\eta^7\eta^8$ must come from amplitude $\tilde{A}^{\mathcal{N}=4}$, which means that $A^{\mathcal{N}=4}$ and $\tilde{A}^{\mathcal{N}=4}$ must be amplitudes of scalars and gluons respectively.

Up to this step, particle contents do match up those in the $\mathcal{N} = 8$ KLT relation. However, when checking the scalar part from $A^{\mathcal{N}=4}$ carefully, we see immediately that it is not the pure scalar theory with cubic vertex interaction. As a matter of fact in $\mathcal{N} = 4$ SYM theory a cubic vertex that links three scalars is forbidden by charge conservation. Thus despite the many similarities shared between KLT relation conjectured by BFW and the KLT relations of the $\mathcal{N} = 8$ theory, the later can not be used to derive former.

Although $\mathcal{N} = 8$ KLT relation does not provide an explanation to the conjecture, in principle, the conjectured KLT relation can be derived from the field theory limit of heterotic string theory as demonstrated in [28] with careful calculations of cocycle part. So far we have not seen a complete derivation along this line and it would be interesting to do so rigorously.

In this paper we prove the KLT relation conjectured by BFW [26] from field theory viewpoint. We will follow similar steps in [9, 10, 11, 12]. To be able to do so first we take special cares on properties of scalar amplitudes.

The structure of this paper is organized as follows: In section 2, we write down the BCFW recursion relation for the scalar theory where the boundary contribution is nonzero. The nonzero boundary contribution will bring many new features in our later discussion. In section 3 we prove the color order reversed identity and $U(1)$ -decoupling identity using BCFW recursion relation. Sections 4 and 5 are devoted to proofs of KK and BCJ relations, where an off-shell fundamental BCJ relation is established. Having all the preparations above, we finally prove the conjectured KLT relation in section 6. There we provide two proofs: one uses the BCFW recursion and the other one, a direct proof based on the off-shell fundamental BCJ relation. A conclusion and discussions are given in the last section. Finally a generalized $U(1)$ -decoupling relation which explains the vanishing of boundary contributions in KK relations is supplied in the Appendix.

2. BCFW recursion relation and Boundary contribution in ϕ^3 theory

In the color-KLT relations proposed in [26], color-dressed pure Yang-Mills amplitudes are expressed as a sum over color-part scalar amplitudes multiplied by color-ordered amplitudes of gluon. The scalar theory considered here contains only a three-point vertex defined by the totally antisymmetric structure constant f^{abc} which satisfies Jacobi identity (we will call it the "color-ordered scalar theory" from now on)

$$\sum_b (f^{a_1 a_2 b} f^{b a_3 a_4} + f^{a_1 a_4 b} f^{b a_2 a_3} + f^{a_1 a_3 b} f^{b a_4 a_2}) = 0. \quad (2.1)$$

We will see that antisymmetry provides enough condition for proving the relations to be discussed in sections 3 and 4, but for BCJ relation, the Jacobi-identity (2.1) is crucial. The Jacobi-identity (2.1), when expressed in terms of three-point amplitudes, reads

$$A_3(1, 2, -P_{12})A_3(-P_{34}, 3, 4) + A_3(1, 4, -P_{14})A_3(-P_{23}, 2, 3) + A_3(1, 3, -P_{13})A_3(-P_{42}, 4, 2) = 0. \quad (2.2)$$

One thing worth mentioning about equation (2.2) is that for the color-ordered scalar theory which contains only cubic vertex, the three-point amplitude is the same for both on-shell and off-shell momenta. This natural relation between off-shell momenta and on-shell momenta for scalar field plays a very important role in many of the properties to be discussed later. One reason for this simplicity is that the wave function of scalar in momentum space is simply a number, 1, which does not depend on momentum.

The BCFW recursion relation of an amplitude is derived from applying Cauchy's theorem to the complex integral

$$\int \frac{dz}{z} A(z) = A(0) + \sum_{\text{poles } z_i} A(z_i) = B, \quad (2.3)$$

where in the integrand a shifted amplitude $A(z)$ is given as an analytic continuation defined by shifting a pair of the legs (i, j) by a light-like 4-vector, $\hat{p}_i = p_i + zq$, $\hat{p}_j = p_j - zq$ with $q^2 = q \cdot p_i = q \cdot p_j = 0$. Generically the integral (2.3) can have boundary contribution B if $A(z)$ does not vanish at infinity and the unshifted amplitude therefore equals boundary contribution minus the sum over residues taken at finite z_i , which assume the forms as cut amplitudes $A_L(z_i) \frac{1}{P^2} A_R(z_i)$ [3, 4]. From the structure of Feynman diagrams we see that when legs (i, j) are not adjacent there will be at least one propagator carrying a factor of $\frac{1}{z}$ bridging between the two shifted legs so that $A(z)$ vanishes at infinity. However boundary terms need to be taken into account when (i, j) are adjacent [6]. For example when the two shifted legs are p_1 and p_n we have

$$B = A_L(n, 1, -P) \frac{1}{s_{n1}} A_R(P, 2, \dots, n-1). \quad (2.4)$$

In the expression (2.4) above, the leg P in $A(P, \dots)$ is off-shell in general. However, as we have mentioned before, being different from vectors and spinors, the scalar wave function does not depend on the momentum and there is a natural analytic continuation from on-shell amplitudes to off-shell amplitude, thus boundary (2.4) with off-shell momentum P does not bother us much.

As a warm up let us perform a consistent check by calculating the same four-point scalar amplitude $A_4(1234)$ from all three possible pairs of shiftings. It is easy to see that boundary contribution arise when we shift $(1, 2)$, and

$$A_4^{(12)}(1, 2, 3, 4) = \frac{\sum_b f^{a_4 a_1 b} f^{b a_2 a_3}}{s_{41}} + \frac{\sum_b f^{a_1 a_2 b} f^{b a_3 a_4}}{s_{12}}, \quad (2.5)$$

where the first term in (2.5) comes from cutting propagator $1/S_{41}$ and the second arises as a boundary term. Exactly the same formula is obtained from shifting $(4, 1)$, whereas in this case the roles played by the two terms in (2.5) are swapped. The $(1, 3)$ -shifting is a bit different since the legs are not adjacent. In this case both terms are given as cuts and we obtain the same result for $A_4^{(13)}$.

We note that in the four-point example here the consistency among three different pairs of shiftings does not lay further constraints on structure of the coupling f^{abc} . The same is not true with gauge theory. In [29], Cachazo and Benincasa have shown that the the same consistency at four-point requires nontrivial Jacobi identity². Also note that highly nontrivial identities derived from consistency requirement imposed by taking different choices of pairs of shiftings and the cancelation of spurious poles, have played important role in recent development of scattering amplitudes of $\mathcal{N} = 4$ theory as emphasized by [30, 31].

3. Color-order reversed relation and $U(1)$ -decoupling relation

Although for gauge theory color-order reversed relation and $U(1)$ -decoupling relation are somewhat trivial, it is not so for color-ordered scalar theory, especially there is no natural " $U(1)$ -decoupling" analog.

First let us consider the color-order reverse relation, which is given by

$$A(1, 2, \dots, n) = (-1)^n A(n, n-1, \dots, 1). \quad (3.1)$$

We prove the relation by induction. The starting point is the three-point amplitude given by

$$A(1, 2, 3) = f^{a_1 a_2 a_3} = -f^{a_3 a_2 a_1} = -A(3, 2, 1), \quad (3.2)$$

We see that equation (3.1) is clearly satisfied by the totally antisymmetric coupling constant f^{abc} . For general n , we expand the amplitude by BCFW recursion relation with $(1, n)$ -shifting as

$$\begin{aligned} A(1, 2, \dots, n) &= \sum_{i=2}^{n-2} \sum_a A(\widehat{1}, 2, \dots, i, -\widehat{P}_{1i}^a) \frac{1}{s_{1i}} A(\widehat{P}_{1i}^a, i+1, \dots, \widehat{n}) \\ &\quad + \sum_a A(n, 1, -P_{1,n}^a) \frac{1}{s_{1,n}} A(P_{1,n}^a, 2, \dots, n-1). \end{aligned} \quad (3.3)$$

where the second line came from boundary contribution and a color a is understood to be summed over. Applying the color-order reversed relation to every sub-amplitude in above expression and recognizing that it is nothing but the BCFW expansion of another amplitude, we get immediately the color order reversed relation

$$\begin{aligned} A(1, 2, \dots, n) &= (-1)^{n+2} \sum_{i=2}^{n-2} \sum_a A(\widehat{n}, n-1, \dots, i+1, \widehat{P}_{1i}^a) \frac{1}{s_{1i}} A(\widehat{P}_{1i}^a, i, \dots, \widehat{1}) \\ &\quad + \sum_a A(n-1, \dots, 2, P_{1,n}^a) \frac{1}{s_{1,n}} A(-P_{1,n}^a, 1, n) \\ &= (-1)^n A(n, n-1, \dots, 1), \end{aligned} \quad (3.4)$$

where we see that only the antisymmetric property of three-point amplitude and the applicability of the BCFW recursion relation with boundary were needed in deriving this result.

²See [32, 33] for generalization to arbitrary n -gluons.

Next let us we move on to the $U(1)$ -decoupling identity given by

$$A(1, 2, 3, \dots, n) + A(1, 3, 2, \dots, n) + \dots + A(1, 3, 4, \dots, n, 2) = 0. \quad (3.5)$$

The simplest $U(1)$ -decoupling identity is for $n = 3$, which is nothing but the color-order reversed identity. The next simplest case for four-point amplitude can be checked directly using the explicit result given in (2.5)

$$\begin{aligned} I &= A_4(1, 2, 3, 4) + A_4(1, 3, 2, 4) + A_4(1, 3, 4, 2) \\ &= \left[\frac{\sum_b f^{a_4 a_1 b} f^{b a_2 a_3}}{s_{41}} + \frac{\sum_b f^{a_1 a_2 b} f^{b a_3 a_4}}{s_{12}} \right] + \left[\frac{\sum_b f^{a_4 a_1 b} f^{b a_3 a_2}}{s_{41}} + \frac{\sum_b f^{a_1 a_3 b} f^{b a_2 a_4}}{s_{13}} \right] \\ &\quad + \left[\frac{\sum_b f^{a_2 a_1 b} f^{b a_3 a_4}}{s_{12}} + \frac{\sum_b f^{a_1 a_3 b} f^{b a_4 a_2}}{s_{13}} \right] \\ &= \frac{\sum_b f^{a_4 a_1 b} f^{b a_2 a_3} + \sum_b f^{a_4 a_1 b} f^{b a_3 a_2}}{s_{41}} + \frac{\sum_b f^{a_1 a_2 b} f^{b a_3 a_4} + \sum_b f^{a_2 a_1 b} f^{b a_3 a_4}}{s_{12}} \\ &\quad + \frac{\sum_b f^{a_1 a_3 b} f^{b a_4 a_2} + \sum_b f^{a_1 a_3 b} f^{b a_2 a_4}}{s_{13}} \\ &= 0, \end{aligned}$$

where again, only the totally anti-symmetric property of coupling constant f^{abc} is used.

For general n we derive an inductive proof via $(1, 2)$ -shifting. For $3 \leq k \leq n-1$, the BCFW-expansion of a general n -point amplitude is (We drop the summation over color index a for simplicity.)

$$\begin{aligned} A(1, 3, \dots, k, 2, k+1, \dots, n) &= \sum_{n \geq j \geq k+1} \sum_{3 \leq i \leq k} A_L(j, j+1, \dots, n, 1, 3, \dots, i, P_{ji}) \frac{1}{P_{ji}^2} A_R(-P_{ji}, i+1, \dots, k, 2, \dots, j-1) \\ &\quad + \sum_{n \geq j \geq k+1} A_L(j, j+1, \dots, n, 1, P_{j1}) \frac{1}{P_{j1}^2} A_R(-P_{j1}, 3, \dots, k, 2, \dots, j-1) \\ &\quad + \sum_{3 \leq i \leq k} A_L(1, 3, \dots, i, P_{1i}) \frac{1}{P_{1i}^2} A_R(-P_{1i}, i+1, \dots, k, 2, \dots, n), \end{aligned} \quad (3.6)$$

where we have isolated in the last two lines terms that contain a cut adjacent to leg 1. There are also two special amplitudes that produce nonzero boundary contributions under BCFW-expansion:

$$\begin{aligned} A(1, 2, 3, \dots, n) &= A_L(1, 2, P) \frac{1}{s_{12}} A_R(-P, 3, 4, \dots, n) \\ &\quad + \sum_{n \geq j \geq 4} A_L(j, j+1, \dots, n, 1, P_{j1}) \frac{1}{P_{j1}^2} A_R(-P_{j1}, 2, 3, \dots, j-1) \end{aligned} \quad (3.7)$$

and

$$\begin{aligned} A(1, 3, 4, \dots, n, 2) &= A_L(2, 1, P) \frac{1}{s_{12}} A_R(-P, 3, 4, \dots, n) \\ &\quad + \sum_{3 \leq i \leq n-1} A_L(1, 3, \dots, i, P_{1i}) \frac{1}{P_{1i}^2} A_R(-P_{1i}, i+1, \dots, n, 2) \end{aligned} \quad (3.8)$$

Having written down BCFW-expansion for every amplitudes, we sum them up and collect various contributions for given cut momentum. There are two contributions for the cut momentum P_{12} coming from (3.7) and (3.8) and we have

$$[A_L(1, 2, P) + A_L(2, 1, P)] \frac{1}{s_{12}} A_R(-P, 3, 4, \dots, n) = 0$$

where the anti-symmetry property of three-point amplitudes was used. For the cut momentum P_{1i} with $3 \leq i \leq n-1$, both (3.6) and (3.8) have contributions as

$$A_L(1, 3, \dots, i, P_{1i}) \frac{1}{P_{1i}^2} \left\{ A_R(-P_{1i}, i+1, \dots, n, 2) + \sum_{i \leq k \leq n-1} A_R(-P_{1i}, i+1, \dots, k, 2, \dots, n) \right\} = 0,$$

where we have used $U(1)$ -decoupling identity for the part inside the big curly bracket by induction. For the cut momentum P_{j1} both (3.6) and (3.7) have contributions as

$$A_L(j, j+1, \dots, n, 1, P_{j1}) \frac{1}{P_{j1}^2} \left\{ A_R(-P_{j1}, 2, 3, \dots, j-1) + \sum_{3 \leq k \leq j-1} A_R(-P_{j1}, 3, \dots, k, 2, \dots, j-1) \right\} = 0$$

where again by the induction, $U(1)$ -decoupling identity has been applied to the part inside the big curly bracket. Finally for the cut momentum P_{ji} with $n \geq j \geq 4$, $3 \leq i \leq n-1$ and $j-i \geq 2$, only (3.6) gives contribution with $i \leq k \leq j-1$, thus we have

$$A_L(j, j+1, \dots, n, 1, 3, \dots, i, P_{ji}) \frac{1}{P_{ji}^2} \sum_{i \leq k \leq j-1} A_R(-P_{ji}, i+1, \dots, k, 2, \dots, j-1) = 0$$

where the induction is applied to the part $\sum_k A_R$.

Before concluding this section, we would like to emphasize again that both the color-order reserved identity and $U(1)$ -decoupling identity used only the anti-symmetric property of coupling constant f^{abc} and the BCFW recursion relation with boundary contribution. Especially the Jacobi identity (2.1) is not needed for these two properties. Also, when we take the external momentum from on-shell to off-shell, these two identities hold too. These facts will be used in our proof of BCJ relation.

4. The KK relation for color-ordered scalar theory

In this section, we will prove that for the color-ordered scalar theory, there is a similar KK-relation found originally in [14] for gauge theory:

$$A_n(\beta_1, \dots, \beta_r, 1, \alpha_1, \dots, \alpha_s, n) = (-1)^r \sum_{\{\sigma\} \in P(O\{\alpha\} \cup O\{\beta\}^T)} A_n(1, \{\sigma\}, n), \quad (4.1)$$

where the sum is over all permutations keeping relative ordering inside the set α and the set β^T (where the T means the reversing ordering of the set β), but allowing all relative ordering between the set α and

β . When there is no element in the $\{\alpha\}$, (4.1) reduces to the color-order reversed relation(3.1), while when there is only one element in the set $\{\beta\}$, (4.1) reduces to the $U(1)$ -decoupling identity(3.5). Thus the color-order reversed relation and the $U(1)$ -decoupling identity are just the special cases of KK relation. If KK-relation is true, above two identities are also true, but the reverse is not guaranteed and it is possible that both identities are true, but the KK-relation is not true³. Since for $n \leq 5$, there is no true KK-relation not covered by the color-order reversed relation and the $U(1)$ -decoupling identity, the starting point of induction proof is checked.

Now we give the general proof of the KK relation by BCFW recursion relation with shifting pair $(1, n)$. The idea will be same as the one used in [18], but there is one major difference: at the left hand side of (4.1), there is no boundary contribution since we have assumed $r \geq 2$, while at the right hand side, each amplitude will give a boundary contribution. Since from the proof given in [18], we see that the pole parts of the left hand side and the right hand side match up, thus for (4.1) to be true, the sum of all boundary contributions must be zero.

Having the above general picture, we repeat some steps given in [18] for self-completion. Using the BCFW recursion relation for scalar field, the amplitude $A_n(\beta_1, \dots, \beta_r, 1, \alpha_1, \dots, \alpha_s, n)$ can be written as (it is worth to notice that there is no boundary contribution)

$$A_n(\beta_1, \dots, \beta_r, 1, \alpha_1, \dots, \alpha_s, n) = \sum_{\text{All splittings}} A_n(\{\beta_L\}, \hat{1}, \{\alpha_L\} | \{\alpha_R\}, \hat{n}, \{\beta_R\}), \quad (4.2)$$

where we have used the short notation $A(\alpha|\beta) \equiv A_L(\alpha, -\hat{P}_\alpha) \frac{1}{P_\alpha^2} A_R(\hat{P}_\alpha, \beta)$. For each splitting $A(\alpha|\beta)$ we can use the KK-relation for $A_L(\alpha, -\hat{P}_\alpha)$ and $A_R(\hat{P}_\alpha, \beta)$ by our induction, thus we have

$$\begin{aligned} & A_n(\beta_1, \dots, \beta_r, 1, \alpha_1, \dots, \alpha_s, n) \\ &= (-1)^{n_{\beta_L} + n_{\beta_R}} \sum_{\text{All splittings } \{\sigma_L \in P(O\{\alpha_L\} \cup O\{\beta_L\}^T)\} \{\sigma_R \in P(O\{\alpha_R\} \cup O\{\beta_R\}^T)\}} \sum A_n(\hat{1}, \{\sigma_L\} | \{\sigma_R\}, \hat{n}) \\ &= (-1)^r \sum_{\{\sigma\} \in P(O\{\alpha\} \cup O\{\beta^T\})} \sum_{\text{All splittings for } \sigma} A_n(\hat{1}, \{\sigma_L\} | \{\sigma_R\}, \hat{n}). \end{aligned} \quad (4.3)$$

It is worth to notice that for the scalar theory with $1, n$ nearby, there is boundary contribution for each amplitude, thus the last line in above equation should be rewritten as

$$\begin{aligned} & A_n(\beta_1, \dots, \beta_r, 1, \alpha_1, \dots, \alpha_s, n) \\ &= (-1)^r \sum_{\{\sigma\} \in P(O\{\alpha\} \cup O\{\beta^T\})} \left\{ A_n(\hat{1}, \{\sigma\}, \hat{n}) - A_3(n, 1, -P_{1,n}) \frac{1}{P_{1,n}^2} A_{n-1}(P_{1,n}, \{\sigma\}) \right\}. \end{aligned} \quad (4.4)$$

Thus for the KK-relation (4.1) to be true, we must show

$$\sum_{\{\sigma\} \in P(O\{\alpha\} \cup O\{\beta^T\})} A_{n-1}(P_{1,n}, \{\sigma\}) = 0. \quad (4.5)$$

³We do not have any example yet and it will be interesting to see if there is an example.

Equation (4.5) is easy to show by using the KK-relation with less number of particles. To see it, assuming the ordering of set α is $\alpha = \{\alpha_1, \dots, \alpha_k\}$ and the set β^T is $\beta^T = \{\beta_1, \dots, \beta_m\}$, then in the sum $\sum_{P(O(\alpha) \cup O(\beta^T))}$ all allowed orderings can be divided into two cases: the first case is that the last element is α_k and the second case is that the last element is β_m . All terms of the first case can be sum up to

$$I_1 = (-)^m A(1, \{\alpha_1, \dots, \alpha_{k-1}\}, \alpha_k, \{\beta_m, \beta_{m-1}, \dots, \beta_1\}) . \quad (4.6)$$

while all terms of the second case can be sum up to

$$I_2 = (-)^{m-1} A(1, \{\alpha_1, \dots, \alpha_{k-1}, \alpha_k\}, \beta_m, \{\beta_{m-1}, \dots, \beta_1\}) . \quad (4.7)$$

It is then easy to see that (4.5) equals to

$$I_1 + I_2 = 0 . \quad (4.8)$$

The proof of (4.5) used the KK-relation, thus it is also true for gauge theory. In fact, as we will explain in the Appendix, it has a very natural physical meaning, i.e., the generalized $U(1)$ -decoupling identity. Also, the momentum P is off-shell in (4.5) does not matter to our proof, i.e., the KK-relation is true with off-shell momentum with natural off-shell continuation.

5. Fundamental BCJ relations of color-dressed scalar amplitudes

In this section we show that the BCJ relation, which was found by Bern, Carrasco and Johansson in [13] for gauge theory, also applies to the color-ordered scalar theory provided that Jacobi identity (2.1) is satisfied by the coupling constant f^{abc} .

To see the reason for introducing Jacobi identity (2.1), let us consider the BCJ relation for four-point amplitudes

$$s_{21} \tilde{A}(1234) + (s_{21} + s_{23}) \tilde{A}(1324) = 0, \quad (5.1)$$

We decorate scalar amplitudes with “tildes” so that they are not to be confused with gluon amplitudes when we discuss KLT relations in later sections. Inserting explicit expressions of four-point amplitudes, and we have

$$\begin{aligned} s_{12} \tilde{A}(1, 2, 3, 4) &= s_{12} \left[\frac{\sum_b f^{a_4 a_1 b} f^{b a_2 a_3}}{s_{41}} + \frac{\sum_b f^{a_1 a_2 b} f^{b a_3 a_4}}{s_{12}} \right], \\ s_{13} \tilde{A}(1, 3, 2, 4) &= s_{13} \left[\frac{\sum_b f^{a_4 a_1 b} f^{b a_3 a_2}}{s_{41}} + \frac{\sum_b f^{a_1 a_3 b} f^{b a_2 a_4}}{s_{13}} \right], \end{aligned} \quad (5.2)$$

The left hand side of equation (5.1) can therefore be written as

$$\begin{aligned} &s_{12} \tilde{A}(1, 2, 3, 4) - s_{13} \tilde{A}(1, 3, 2, 4) \\ &= \frac{s_{12} \sum_b f^{a_4 a_1 b} f^{b a_2 a_3} - s_{13} \sum_b f^{a_4 a_1 b} f^{b a_3 a_2}}{s_{41}} + \sum_b f^{a_1 a_2 b} f^{b a_3 a_4} - \sum_b f^{a_1 a_3 b} f^{b a_2 a_4}, \end{aligned} \quad (5.3)$$

where we have used conservation of momentum to rewrite $(s_{12} + s_{32}) = -s_{13}$. To carry out rest of the calculation we note that Jacobi identity is essential in combining the last two terms, giving

$$\begin{aligned}
& s_{12}\tilde{A}(1, 2, 3, 4) - s_{13}\tilde{A}(1, 3, 2, 4) \\
&= \frac{s_{12} \sum_b f^{a_4 a_1 b} f^{b a_2 a_3} - s_{13} \sum_b f^{a_4 a_1 b} f^{b a_3 a_2}}{s_{41}} - \sum_b f^{a_1 a_4 b} f^{b a_2 a_3} \\
&= \frac{s_{12} \sum_b f^{a_4 a_1 b} f^{b a_2 a_3} - (s_{13} + s_{14}) \sum_b f^{a_4 a_1 b} f^{b a_3 a_2}}{s_{41}} \\
&= \frac{s_{12} \sum_b f^{a_4 a_1 b} f^{b a_2 a_3} + s_{12} \sum_b f^{a_4 a_1 b} f^{b a_3 a_2}}{s_{41}} \\
&= 0.
\end{aligned} \tag{5.4}$$

Since KK-relation is satisfied by the color-ordered scalar theory, to show the general BCJ relation is also satisfied we only need to prove the following fundamental BCJ relation

$$0 = I_n = s_{21}A(1, 2, \dots, n) + \sum_{i=3}^{n-1} (s_{21} + \sum_{t=3}^i s_{2t})A(1, 3, \dots, i, 2, i+1, \dots, n-1, n). \tag{5.5}$$

To prove (5.5) we deform I_n by shifting the pair of legs $(1, n)$, yielding

$$I_n(z) = s_{2\hat{1}}\tilde{A}(\hat{1}23 \dots \hat{n}) + (s_{2\hat{1}} + s_{23})\tilde{A}(\hat{1}32 \dots \hat{n}) + \dots (s_{2\hat{1}} + \dots s_{2,n-1})\tilde{A}(\hat{1}3 \dots n-1, \hat{n}) \tag{5.6}$$

and then consider the following contour integration

$$B_n \equiv \int \frac{dz}{z} I_n(z) = I_n(0) + \sum_{\text{poles } z_i} I_n(z_i) \tag{5.7}$$

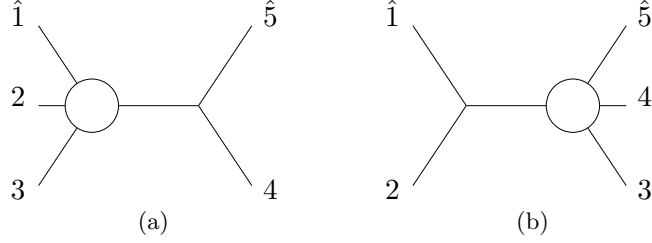
The contour integration can be done by two different ways. The first way is to integrate around the pole at infinity and the result is the boundary contribution B_n . The second way is to integrate along large enough circle around point $z = 0$ and we obtain the right hand side of (5.7), which includes two parts, i.e., the I_n part and the pole parts.

In the proof of BCJ relation for gauge theory, the $\frac{1}{z^2}$ behavior of gauge amplitudes with nonadjacent shifted legs is crucial to derive $B_n = 0$ in [18]. Then by the induction, we can show the pole part is zero, from there we conclude that $I_n = 0$. For color-ordered scalar theory, it is again easy to show the pole part to be zero by induction, thus to prove $I_n = 0$ we must show $B_n = 0$. However, for the scalar theory, the large z behavior of amplitudes with non-nearby shifting pair is $\frac{1}{z}$, thus many boundary terms are expected. The cancelation of these boundary terms relies crucially on the Jacobi identity as we will see shortly.

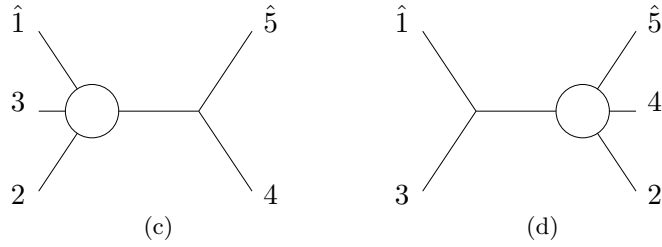
Having the above general picture, we will present our proof in two steps: First we will show that the pole part is zero, and then we will show that the boundary contribution is zero.

5.1 The pole part

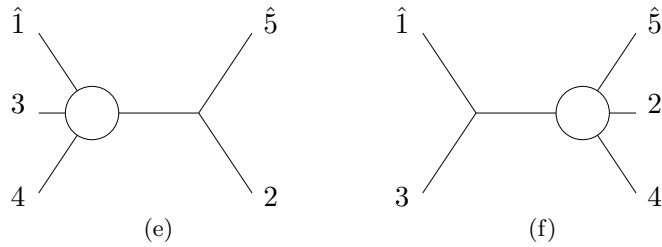
In this part of the discussion we consider the contributions from pole part. Before going to general n , let us check the example when $n = 5$. The pole part contribution from $\tilde{A}(\hat{1}234\hat{5})$ is given by figures (1(a)) and (1(b)).



Similarly, the pole part from $\tilde{A}(\hat{1}324\hat{5})$ is given by (1(c)) and (1(d))



and finally the pole part from $\tilde{A}(\hat{1}342\hat{5})$ is given by (1(e)) and (1(f)).



It is easy to see that a factor $\frac{1}{P_{45}} \tilde{A}(\hat{5}, 4, -P_{45})$ is shared between graphs 1(a) and 1(c), thus the combination of these two contributions gives

$$s_{2\hat{1}} \tilde{A}(\hat{1}, 2, 3, -P_{123}) + (s_{2\hat{1}} + s_{23}) \tilde{A}(\hat{1}, 3, 2, -P_{123}), \quad (5.8)$$

which is zero using 4-point BCJ relation, as was verified explicitly. Again using 4-point BCJ relation we see that the sum of graphs 1(d) and 1(f) is zero after using momentum conservation to rewrite $(s_{2\hat{1}} + s_{23}) = -(s_{24} + s_{2\hat{5}})$ and $(s_{2\hat{1}} + s_{23} + s_{24}) = -s_{2\hat{5}}$. Finally the graphs 1(b) and 1(e) vanish individually

because the shifted internal momenta become null at pole, i.e., from the BCJ relation of 3-point amplitude, $s_{12}\tilde{A}(1, 2, 3) = 0$.

Generalization of the 5-point example to n-point BCJ relation is straightforward. Pole contributions from (5.6) fall into one of following two categories: Those with leg 2 attached to the sub-amplitude on the left hand side of the BCFW-expansion and those to the sub-amplitude on the right hand side of the BCFW-expansion. Graphs with leg 2 attached to the left hand side with split momentum $P_{12\dots i}$ add up to give zero from $i + 1$ -point BCJ relation,

$$\begin{array}{c} \hat{1} \\ s_{2\hat{1}} 2 \\ \vdots \\ i \end{array} \text{---} \text{---} \begin{array}{c} \hat{n} \\ \vdots \\ i+1 \end{array} + \dots + (s_{2\hat{1}} + \dots + s_{2i}) \begin{array}{c} \hat{1} \\ \vdots \\ i \\ 2 \end{array} \text{---} \text{---} \begin{array}{c} \hat{n} \\ \vdots \\ i+1 \end{array} = 0$$

Flipping kinematic factors using momentum conservation as was done in the 5-point example, we see that graphs with leg 2 attached to the right hand side with split momentum $P_{13\dots i}$ also add up to zero.

$$-(s_{2,i+1} + \dots + s_{2\hat{n}}) \begin{array}{c} \hat{1} \\ \vdots \\ i \end{array} \text{---} \text{---} \begin{array}{c} \hat{n} \\ \vdots \\ i+1 \\ 2 \end{array} + \dots - s_{2\hat{n}} \begin{array}{c} \hat{1} \\ \vdots \\ i \end{array} \text{---} \text{---} \begin{array}{c} \hat{n} \\ 2 \\ \vdots \\ i+1 \end{array} = 0$$

Thus, using the BCJ relation of \tilde{A}_m for $m \leq n$, we have shown that the pole part of (5.6) is zero.

5.2 the generalized fundamental BCJ relation with one off-shell momentum

Before showing that the boundary terms in the integral (5.7) cancel, here we present a generalized scalar BCJ relation to be used later in the proof, where the leg n carries an off-shell momentum (but still color-ordered):

$$\begin{aligned}
 & s_{21}\tilde{A}(123\dots; n) + (s_{21} + s_{23})\tilde{A}(132\dots; n) + \dots (s_{21} + \dots s_{2,n-1})\tilde{A}(13\dots n-1, 2; n) \\
 & = -\sum_{color\ c} p_n^2 f^{n2c} \frac{1}{P_c^2} \tilde{A}(1, 3\dots n-1; c) \quad (5.9)
 \end{aligned}$$

It is clear that equation (5.9) is identical to the fundamental BCJ relation (5.5) when p_n is on-shell. The relation is easily illustrated in terms of Figure 1

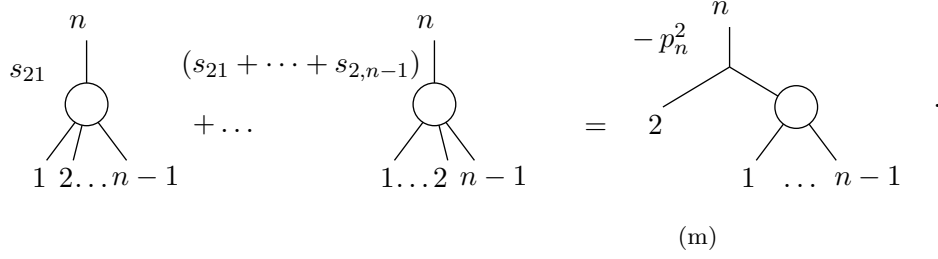


Figure 1: The graph representation of off-shell BCJ relation.

We prove the off-shell relation (5.9) by induction. At 3-point, equation (5.9) simply restates the algebraic relation between momenta p_1 , p_2 and p_3

$$s_{21}\tilde{A}(12; 3) = p_3^2 \tilde{A}(12; 3) \quad (5.10)$$

where we have used the degenerated case $A(3; c) = \delta_3^c P_c^2$ and the antisymmetric property $f^{321} = -f^{123}$.

To get a clearer picture of how a proof is constructed when generalized to n -points, let us consider applying equation (5.9) to 4-point amplitudes. At 4-points, the left side of the equation is given by

$$s_{21}\tilde{A}(123; 4) + (s_{21} + s_{23})\tilde{A}(132; 4), \quad (5.11)$$

which can as well be represented by a summation over the six graphs from Figure2 (2(a) to 2(f)). These figures are obtained by the Feynman diagram expansion multiplied by the corresponding kinematic factors s_{21} and s_{23} .

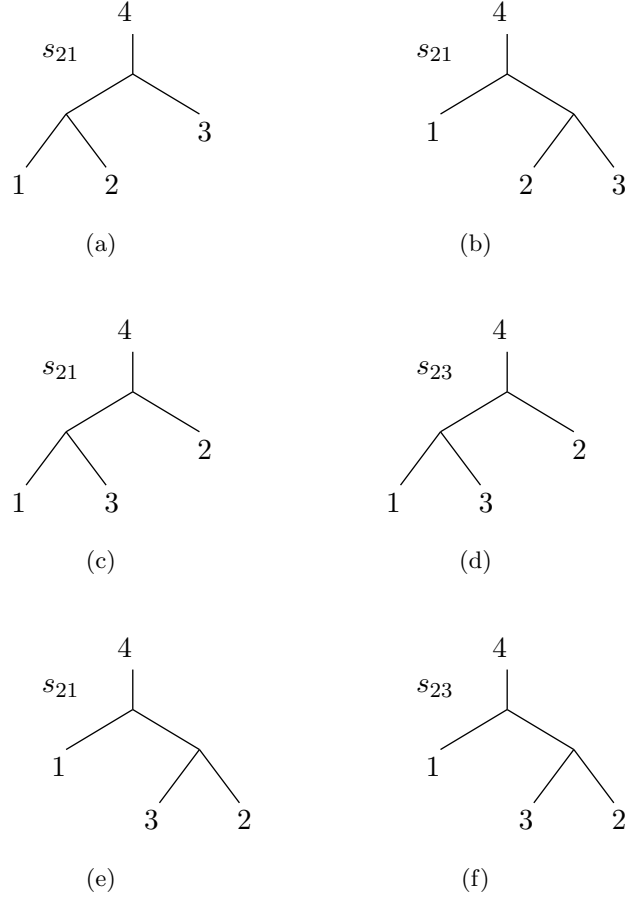


Figure 2: The boundary contributions of four-point BCJ relation.

We see that graphs 2(b) and 2(e) cancel each other as the result of the antisymmetry of the vertex connecting legs 2 and 3. In graph 2(a) and graph 2(f), the internal lines are canceled by kinematic factors s_{21} and s_{23} respectively, thus we can combine them using Jacobi identity to get Figure 3(c):

$$\begin{array}{c}
 \begin{array}{c} 4 \\ | \\ s_{21} \\ \diagup \quad \diagdown \\ 1 \quad 2 \quad 3 \end{array}
 \quad + \quad
 \begin{array}{c} 4 \\ | \\ s_{23} \\ \diagup \quad \diagdown \\ 1 \quad 3 \quad 2 \end{array}
 \quad = \quad
 \begin{array}{c} 4 \\ | \\ s_{31} \\ \diagup \quad \diagdown \\ 1 \quad 3 \quad 2 \end{array}
 \end{array}
 \quad (c)$$

Figure 3: The combination of (a) and (f) of Figure 2.

Since the same tree structure is shared between 2(c), 2(d) and 3(c), the summation of these three graphs is then given by the sum over the attached kinematic factors $(s_{21} + s_{23} + s_{31})$, multiplied by the

common factors represented by the tree, yielding

$$(s_{21} + s_{23} + s_{31}) \begin{array}{c} 4 \\ | \\ \diagup \quad \diagdown \\ 1 \quad 3 \quad 2 \end{array} = -p_4^2 \begin{array}{c} 4 \\ | \\ 2 \quad \diagup \quad \diagdown \\ \quad 1 \quad 3 \end{array}$$

Figure 4: The off-shell BCJ relation of four-point amplitude.

from which we arrive at the graphical representation of $-p_4^2 f^{42c} \frac{1}{P_c^2} \tilde{A}(13; c)$ as was claimed in (5.9).

Proving n -point off-shell relations by induction

For general n -point relation we note that amplitudes on the left side of the off-shell relation (5.9) can be reorganized according to how the rest of the external legs are connecting to leg n (see Figure 5). For example, for amplitude $A(1, 3, \dots, i, 2, i+1, \dots, n)$, all Feynman diagrams will be of the form drawn in Figure (5(a)) where legs $1, 3, \dots, k$ constitute a subtree connecting to leg n through the left propagator and $k, \dots, n-1$ constitute another subtree connecting to leg n through the right propagator. In these diagrams leg 2 can be inserted into either the left subtree or into the right subtree.

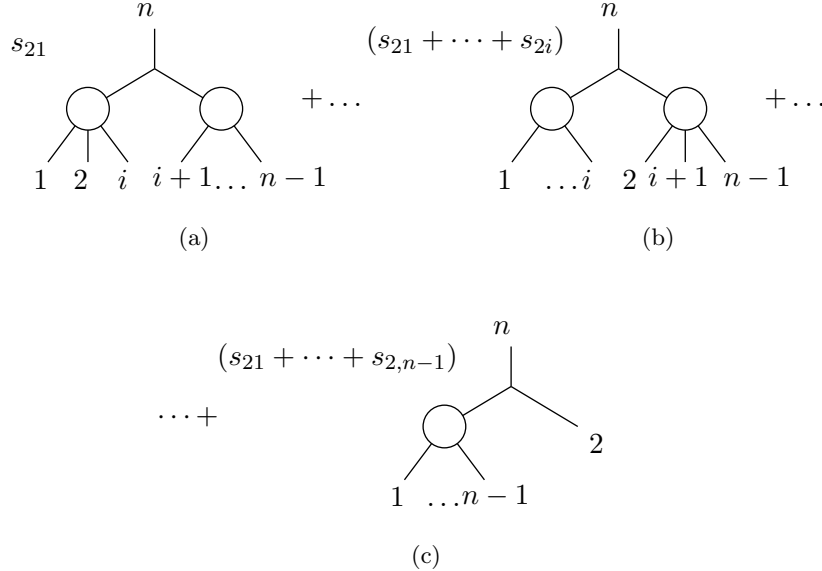


Figure 5: The Feynman diagrams organized according to the vertex leg n attached.

Now we consider the type of Feynman diagrams, where legs $1, 3, 4, \dots, i$ are at the left subtree and $i+1, \dots, n-1$ are at the right subtree while leg 2 is inserted freely into all possible positions (the degenerated

case given in Figure (5(c)) is not included). When leg 2 is at the left subtree (see Figure (5(a))), the sum over these left sub-trees with the kinematic factors is nothing but the form (5.9) with less number of particles, thus by the induction, the result can be represented by the Figure (6(a)) using the graphical identity Figure 1.

When leg 2 is at the right subtrees (the type given by Figure (5(b))), things will be a little bit complicated. First we divide the kinematic factor $(s_{21} + \dots + s_{2i} + s_{2(i+1)} + \dots + s_{2k})$ into two parts: the common factor $(s_{21} + \dots + s_{2i})$ and the remaining factor $(s_{2(i+1)} + \dots + s_{2k})$ when leg 2 is right behind leg k . For the common factor $(s_{21} + \dots + s_{2i})$, when we sum up all possible insertions of leg 2 at the right subtree, we get zero by using the $U(1)$ -decoupling equation (as we have remarked, it is true even with one leg off-shell). For the remaining factor $(s_{2(i+1)} + \dots + s_{2k})$, the sum is again in the form of (5.9), then by the induction we obtain the Figure (6(b)) using the Figure 1.

After the substitutions we are left with Figures (5(c)), (6(a)) and (6(b)) multiplied by kinematic factor $\sum_{j=1}^{n-1} s_{2j}$, $(p_1 + p_3 + \dots + p_i)^2 \equiv P^2$ and $(p_{i+1} + p_{i+2} + \dots + p_{n-1})^2 \equiv Q^2$ respectively. If we effectively regard the subtree containing legs $1, 3, \dots, i$ as a new leg carrying momentum P and the subtree containing legs $i+1, \dots, n-1$ as another new leg carrying momentum Q , Figure (6(a)) with factor P^2 and Figure (6(b)) with factor Q^2 can be combined to give Figure (6(c)) with factor $-(P+Q)^2$ using Jacobi identity. Thus we see clearly why the Jacobi identity is crucial for the BCJ relation to be true. As a last step we combine Figure (5(c)) and (6(c)). Using the anti-symmetric property, algebraically combining kinematic factors $-(P+Q)^2 + 2k_2 \cdot k_n = -p_n^2$, and then summing over i to obtain an amplitude $\tilde{A}(1, 3, \dots, n-1; c)$, we arrive at exactly the same result as was claimed in (5.9).

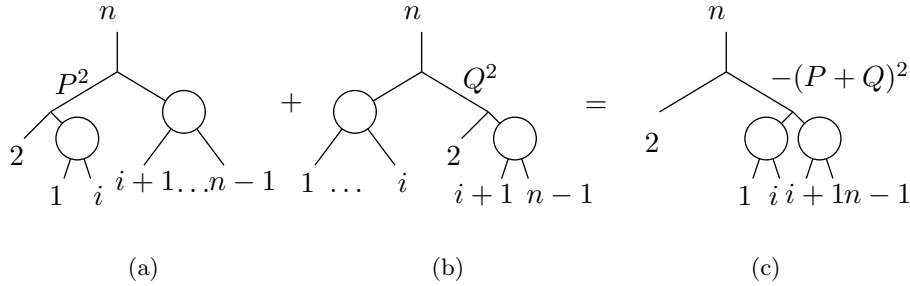


Figure 6: The simplification of the left and right subtrees in Figure 5.

In fact, the proof of (5.9) reduces to a proof of fundamental BCJ relation given in (5.5) when we take p_n on-shell. In other words, we have given a diagrammatic proof of BCJ relation. We shall see in the following section that the cancelation of boundary terms in our original proof of BCJ relation via BCFW recursion also requires the identity (5.9).

5.3 Boundary terms

As we have shown, for the color-ordered scalar theory, there are boundary contributions from the contour $\oint \frac{dz}{z} I_n(z)$. These boundary contributions come from two places when we write $s_{21}(z) = s_{21} + zs_{q1}$: the

first place is boundary contribution from amplitudes with $(1, n)$ nearby and the second place is zs_{q1} with $\frac{1}{z}$ part of amplitudes.

Contributions from the first place is given by $A(n, 1, -P_{n1})\frac{1}{P_{n1}^2}$ multiplying with the following factor

$$s_{21}A(P_{n1}, 2, 3, \dots, n-1) + (s_{21} + s_{23})A(P_{n1}, 3, 2, 4, \dots, n-1) + \dots + (s_{21} + \sum_{j=3}^{n-1} s_{j2})A(P_{n1}, 3, 4, \dots, n-1, 2) .$$

In the equation above, terms carrying the same factor s_{21} sum up to give zero from $U(1)$ -decoupling identity, while the remaining terms assume the form of (5.9), which we replace by

$$-\sum_{a,c} A(n, 1, -P_{n1}^a) f^{a2c} A(3, 4, \dots, n-1; P_{n12}^c) , \quad (5.12)$$

Graphically this is represented by Figure 7 (7(c)).

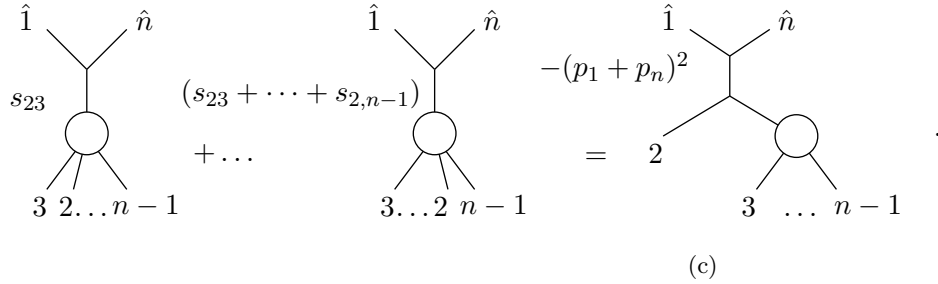


Figure 7: The boundary contribution of first source.

Contributions from the second place come from diagrams that have only one propagator between the shifted legs $\hat{1}$ and \hat{n} , such as the one illustrated in Figure 8 (8(a)). More explicitly, the left cubic vertex has leg 1, propagator with left subtree containing legs $3, \dots, i$, and the propagator connecting $1, n$, while the right cubic vertex has leg n , propagator with left subtree containing some legs $i+1, \dots, 2$, and the propagator connecting $1, n$. Thus except the two degenerated cases given by Figure (8(a)) and (8(b)), the general expression will have the form

$$\left\{ A(1, -P_L, -P_L - p_1) \frac{1}{P_L^2} A_L(P_L, 3, \dots, i) \right\} \times \left\{ A(P_L + p_1, -P_R, n) \frac{1}{P_R^2} A_R(P_R, i+1, \dots, n-1) \right\} \quad (5.13)$$

where leg 2 can be inserted at any allowed position in A_L and A_R . It is easy to see that the sum over all insertions of leg 2 in A_L and A_R gives zero by $U(1)$ -decoupling identity.

From the above analysis, we see that sum of boundary contributions reduces to three terms given by Figure (7(c)), (8(a)) and (8(b)). If we take the subtree made by leg $3, 4, \dots, n-1$ as a new leg $P_{3(n-1)}$, then the sum of the three diagrams are nothing but the three terms that appear in Jacobi identity and we have a complete cancelation.

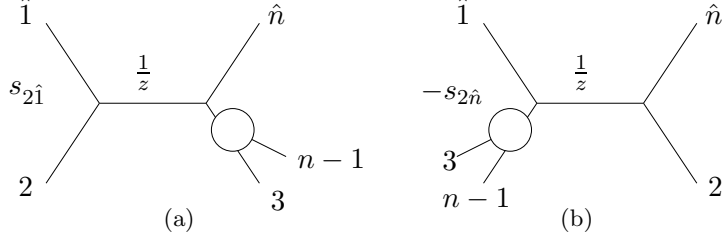


Figure 8: The two degenerated of boundary contributions from second source.

6. Field theory proof of color KLT relations

After the long preparation above, finally in this section we present a pure field theory proof of the KLT relation proposed in [26]. Having shown that the KK-relation and BCJ relation are also true for color-ordered scalar theory, the proof of the proposed KLT relation will be much similar to the one for the KLT relation between gravity and gauge theory in [9, 10, 11, 12], where more details can be found.

6.1 The KLT relation

First we need to present the KLT relations. To do so we introduce following function \mathcal{S} given by

$$\mathcal{S}[i_1, \dots, i_k | j_1, j_2, \dots, j_k]_{p_1} = \prod_{t=1}^k (s_{i_t 1} + \sum_{q>t}^k \theta(i_t, i_q) s_{i_t i_q}), \quad (6.1)$$

where $\theta(i_t, i_q)$ is zero when pair (i_t, i_q) has the same ordering at both set \mathcal{I}, \mathcal{J} and otherwise, it is one. With above definitions, the KLT relation given originally in [34] can be written as

$$M_n = (-)^{n+1} \sum_{\sigma \in S_{n-3}} \sum_{\alpha \in S_j} \sum_{\beta \in S_{n-3-j}} A(1, \{\sigma_2, \dots, \sigma_j\}, \{\sigma_{j+1}, \dots, \sigma_{n-2}\}, n-1, n) \mathcal{S}[\alpha(\sigma_2, \dots, \sigma_j) | \sigma_2, \dots, \sigma_j]_{p_1} \\ \times \mathcal{S}[\sigma_{j+1}, \dots, \sigma_{n-2} | \beta(\sigma_{j+1}, \dots, \sigma_{n-2})]_{p_{n-1}} \tilde{A}(\alpha(\sigma_2, \dots, \sigma_j), 1, n-1, \beta(\sigma_{j+1}, \dots, \sigma_{n-2}), n) \quad (6.2)$$

where $j = [n/2 - 1]$ is a fixed number determined by n and the two subscripts p_1, p_{n-1} of two \mathcal{S} -functions are also important to distinguish. However, using the BCJ relation we can reduce or increase j and we reach following two more symmetric formula

$$M_n = (-)^{n+1} \sum_{\sigma, \tilde{\sigma} \in S_{n-3}} A(1, \sigma(2, n-2), n-1, n) \mathcal{S}[\tilde{\sigma}(2, n-2) | \sigma(2, n-2)]_{p_1} \tilde{A}(n-1, n, \tilde{\sigma}(2, n-2), 1) \quad (6.3)$$

as well as

$$M_n = (-)^{n+1} \sum_{\sigma, \tilde{\sigma} \in S_{n-3}} A(1, \sigma(2, n-2), n-1, n) \mathcal{S}[\sigma(2, n-2) | \tilde{\sigma}(2, n-2)]_{p_{n-1}} \tilde{A}(1, n-1, \tilde{\sigma}(2, n-2), n) \quad (6.4)$$

Furthermore, a manifestly $(n-2)!$ symmetric KLT formula was found in [9]

$$M_n = (-)^n \sum_{\gamma, \beta} \frac{\tilde{A}(n, \gamma(2, \dots, n-1), 1) \mathcal{S}[\gamma(2, \dots, n-1) | \beta(2, \dots, n-1)]_{p_1} A(1, \beta(2, \dots, n-1), n)}{s_{123 \dots (n-1)}} \quad (6.5)$$

as well as its dual form

$$M_n = (-)^n \sum_{\beta, \gamma} \frac{A(1, \beta(2, \dots, n-1), n) \mathcal{S}[\beta(2, \dots, n-1) | \gamma(2, \dots, n-1)]_{p_n} \tilde{A}(n, \gamma(2, \dots, n-1), 1)}{s_{2\dots n}} \quad (6.6)$$

In [35], it was shown explicitly how to regularize the above expression and the equivalence between the new $(n-2)!$ form (6.5) and old $(n-3)!$ form (6.3) was proved.

One important property of function \mathcal{S} is

$$0 = I \equiv \sum_{\alpha \in S_k} \mathcal{S}[\alpha(i_1, \dots, i_k) | j_1, j_2, \dots, j_k]_{p_1} A(k+2, \alpha(i_1, \dots, i_k), 1) . \quad (6.7)$$

or the one obtained by color order reversing

$$0 = I \equiv \sum_{\alpha \in S_k} \mathcal{S}[j_k, \dots, j_1 | \alpha]_{p_1} A(1, \alpha, k+2) . \quad (6.8)$$

6.2 The proof of KLT relation for gauge theory

Now we prove that when A is color-ordered tree level amplitude of gauge theory and \tilde{A} is the color-ordered scalar theory discussed in previous sections, the M_n is nothing but the total scattering amplitude \mathcal{A}_n of gauge theory as conjectured in [26]. The idea of our proof will be the following. With BCFW-deformation of $(1, n)$ we have

$$B_1 = \oint \frac{dz}{z} \mathcal{A}_n(z) = \mathcal{A}_n(z=0) - \sum_{\alpha} \mathcal{A}_L \frac{1}{P_{\alpha}^2} \mathcal{A}_R, \quad (6.9)$$

$$B_2 = \oint \frac{dz}{z} M_n(z) = M_n(z=0) + \sum_{poles} \text{Res} \left(\frac{M_n(z)}{z} \right), \quad (6.10)$$

If $B_1 = 0 = B_2$ and we can use the induction to show that $-\sum_{\alpha} \mathcal{A}_L \frac{1}{P_{\alpha}^2} \mathcal{A}_R = \sum_{poles} \text{Res} \left(\frac{M_n(z)}{z} \right)$, then we have shown $\mathcal{A}_n = M_n$.

The vanishing of B_1 is obvious for gauge theory. For B_2 , let us check the formula (6.4). First the kinematic factor $\tilde{\mathcal{S}}$ is independent of z . Secondly, for nearby $1, n$, the gauge part contributes a $\frac{1}{z}$ and the scalar part, a $\frac{1}{z^0}$, thus the overall behavior will be $\frac{1}{z}$ and we get vanishing boundary contribution B_2 .

Having established the vanishing of boundary contributions B_1, B_2 , we will focus on the pole parts. As in [9, 10, 11, 12], we will show inductively that the BCFW-expansion of M_n gives precisely the BCFW-expansion of full gluon amplitudes. The starting point, i.e., 3-point KLT relation, is easy to check since the scalar amplitude is simply a structure constant

$$M_3(1^a, 2^b, 3^c) = f^{abc} A_3(1, 2, 3). \quad (6.11)$$

For general M_n , by the full symmetry, we can consider a representative cut $P_{123\dots k}$ with $k = 2, \dots, n-2$ and other cuts will be easily related by permutations. For this cut, pole structure in M_n can be further divided into following three categories as in [9, 10, 11, 12]:

- (i) The pole only shows up in scalar amplitude \tilde{A}_n .
- (ii) The pole only shows up in gluon amplitude A_n .
- (iii) Both \tilde{A}_n and A_n contain pole $1/s_{12\dots k}$.

Residues in (i) are given by cutting the scalar amplitude and multiplying $\tilde{\mathcal{S}}$ and A_n with legs 1 and n shifted.

$$\sum_{\sigma, \tilde{\sigma}, \alpha} \frac{\tilde{A}(n-1, \hat{n}, \tilde{\sigma}_{k+1, n-2}, \hat{P}) \tilde{A}(-\hat{P}, \alpha_{2, k}, \hat{1})}{s_{12\dots k}} \mathcal{S}[\tilde{\sigma}_{k+1, n-2} \alpha_{2, k} | \sigma_{2, n-2}]_{\hat{P}_1} A_n(\hat{1}, \sigma_{2, n-2}, n-1, \hat{n}) \quad (6.12)$$

With this configuration, a factor of $\mathcal{S}[\alpha_{2, k} | \rho_{2, k}]_{\hat{P}_1}$ can be extracted from the function $\mathcal{S}[\tilde{\sigma}_{k+1, n-2} \alpha_{2, k} | \sigma_{2, n-2}]_{\hat{P}_1}$. Using (6.7) we find

$$\sum_{\alpha} \tilde{A}(-\hat{P}, \alpha_{2, k}, \hat{1}) \mathcal{S}[\alpha_{2, k} | \rho_{2, k}] = 0. \quad (6.13)$$

Residues in (ii) vanish for the same reason. In the case (iii) the special ordering of A, \tilde{A} allows \mathcal{S} to factorize as the following

$$\mathcal{S}[\tilde{\sigma}_{k+1, n-2} \alpha_{2, k} | \beta_{2, k} \sigma_{k+1, n-2}]_{\hat{P}_1} = \mathcal{S}[\alpha_{2, k} | \beta_{2, k}] \times \mathcal{S}[\tilde{\sigma}_{k+1, n-2} | \sigma_{k+1, n-2}]_{\hat{P}}, \quad (6.14)$$

thus the residue can be written as

$$\begin{aligned} & \frac{1}{s_{12\dots k}} \sum_{h, color} \sum_{\alpha, \beta} \frac{\tilde{A}(-\hat{P}, \alpha_{2, k}, \hat{1}) \mathcal{S}[\alpha_{2, k} | \beta_{2, k}]_{\hat{P}_1} A(\hat{1}, \beta_{2, k}, -\hat{P}^h)}{s_{12\dots k}} \\ & \times \sum_{\sigma, \tilde{\sigma}} \tilde{A}(n-1, \hat{n}, \tilde{\sigma}_{k+1, n-2}, \hat{P}) \mathcal{S}[\tilde{\sigma}_{k+1, n-2} | \sigma_{k+1, n-2}]_{\hat{P}} A(-\hat{P}^h, \sigma_{k+1, n-2}, n-1, \hat{n}). \end{aligned} \quad (6.15)$$

The residue is nothing but the products of sub-amplitudes M_{k+1} and M_{n-k+1} using Eq. (6.3) to Eq. (6.6)

$$\sum_h \frac{M_{k+1}(\hat{1}, 2, \dots, k, -\hat{P}^h) M_{n-k+1}(k+1, \dots, \hat{P}^{-h})}{s_{12\dots k}}. \quad (6.16)$$

Thus we have shown that the pole part of (6.10) does match up with the pole part of (6.9) by induction and we have completed the proof ⁴.

6.3 Another proof by off-shell BCJ relations

For the gauge KLT relation, there is another direct proof using the generalized fundamental BCJ relations discussed in section 5.2. To start with, let us consider following analogous sum appearing in the standard BCJ relations

$$\sum_{\{i\} \in S_{n-2}} \mathcal{S}[2, 3, \dots, n-1 | i_2, i_3, \dots, i_{n-1}] \tilde{A}_n(1, i_2, i_3, \dots, i_{n-1}; n) \quad (6.17)$$

⁴There is a technical point in our proof. For the cut without p_{n-1} it is better to use the formula Eq. (6.3), while for cut having p_{n-1} it is better to use the formula Eq. (6.4).

where as in the previous section we use $\tilde{A}_n(1 \dots n)$ to denote an n -point scalar amplitude and a semicolon is used to emphasize that leg n is allowed to be off-shell. Using the definition of function \mathcal{S} we see that the sum in (6.17) can be written as

$$\begin{aligned} & \sum_{\{j\} \in S_{n-3}} \mathcal{S}[3, \dots, n-1 | j_3, \dots, j_{n-1}] \\ & \times \left[s_{21} A_n(1, 2, j_3 \dots j_{n-1}; n) + (s_{21} + s_{2j_3}) \tilde{A}_n(1, j_3, 2, \dots, j_{n-1}; n) + \dots \right] \\ & = -\frac{p_n^2}{p_{n2}^2} f^{n2c} \sum_{\{i\} \in S_{n-3}} \mathcal{S}[3, \dots, n-1 | j_3, \dots, j_{n-1}] \tilde{A}_{n-1}(1, j_3, \dots; c). \end{aligned} \quad (6.18)$$

where $\{j\}$ is the set defined by deleting leg 2 from the set $\{i\}$ and in the last line we have used (5.9). The sum over \mathcal{S} can be done similarly and we obtain

$$(-)^2 \frac{p_n^2}{p_{n2}^2} f^{n2c} \frac{p_{2n}^2}{p_{n23}^2} f^{c3c_1} \sum_{\{i\} \in S_{n-4}} \mathcal{S}[4, \dots, n-1 | j_3, \dots, j_{n-1}] \tilde{A}_{n-1}(1, j_4, \dots; c_1) \quad (6.19)$$

Repeatedly reducing the number of legs contained in the amplitude one by one, we finally arrive at

$$\sum_{\{i\} \in S_{n-3}} \mathcal{S}[2, 3, \dots, n-1 | \beta_{2,n-1}] \tilde{A}_n(1, \beta_{2,n-1}; n) = (-)^{n-2} p_n^2 f^{n2c} f^{c3e} \dots f^{e,(n-1),1}, \quad (6.20)$$

The result (6.20) can be represented graphic as Figure 9.

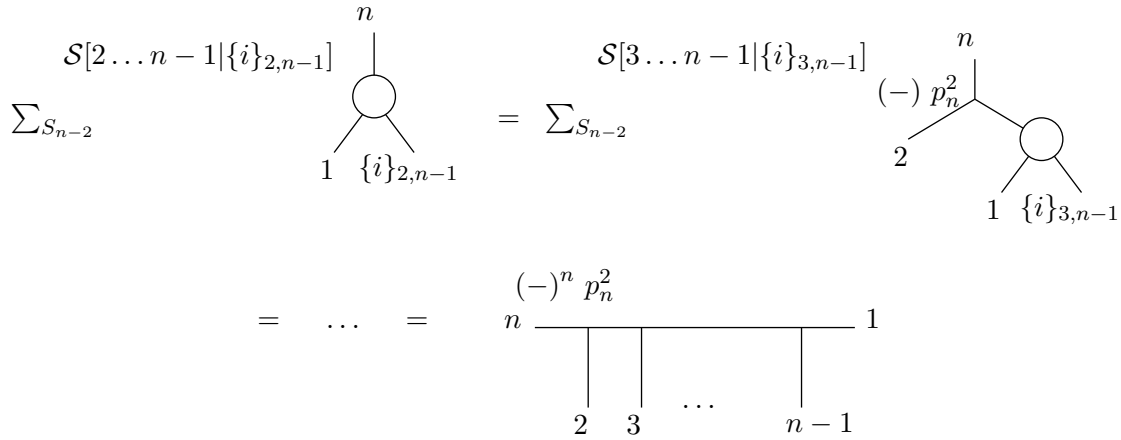


Figure 9: Graphs obtained by repeated substitutions using off-shell fundamental BCJ relations

Now the proof of gluon KLT relations can be obtained directly from application of the off-shell relations (6.20) to following $(n-2)!$ symmetric form

$$M_n = (-1)^n \sum_{\gamma, \beta} \frac{A_n(n, \gamma_{2,n-1}, 1) \mathcal{S}[\gamma_{2,n-1} | \beta_{2,n-1}]_{p_1} \tilde{A}_n(1, \beta_{2,n-1}, n)}{s_{12 \dots, n-1}}, \quad (6.21)$$

and we arrive at the familiar expression discovered by Del Duca, Dixon and Maltoni[17] which expresses a gluon total amplitude as products of structure constants multiplied by color-ordered amplitudes

$$M_n = \sum_{\gamma} f^{n,i_2,c} f^{c,i_3,e} \dots f^{e,i_{n-1},1} A_n(n, i_2, \dots i_{n-1}, 1). \quad (6.22)$$

7. Conclusion

In this paper, we systematically study the color-ordered scalar theory and prove the color-order reversed relation, $U(1)$ -decoupling identity, KK and BCJ relations for this field. In our proof, the BCFW on-shell recursion relation with nontrivial boundary contributions plays the central role. Having established the KK and BCJ relations, we prove the gauge KLT conjecture made in [26], where the full gauge amplitude can be factorized as the product of color-ordered gauge amplitude and the color-ordered scalar amplitude. A byproduct in our proof is the generalized fundamental BCJ relation with one leg off-shell and using it, both BCJ relation and KLT relation can be derived directly.

There are a few things worth to notice. As we have mentioned, in the proof of BCJ relation, the generalized fundamental BCJ relation with one leg off-shell is founded and using it (6.20) gives the direct relation between color and $\mathcal{S}\tilde{A}$ which is not obvious at all. Recently there is a nice suggestion about the dual form of gauge theory given in [36]. If we can find some generalized fundamental BCJ relation with one leg off-shell in the gauge theory, then we maybe possible to give a good construction of dual color factor n_i as discussed in [36].

Another interesting thing is generalized KLT relations conjectured in [26] besides the one studied here, such as the tree amplitudes with gluons coupled to gravitons can be factorized into pure-gluon color-ordered amplitudes and amplitudes with scalar coupled to gluons, or be factorized into fermion pair coupled to gluons. KLT relation has also be found for those amplitudes with a massive graviton exchange when we factorize external gluons into two quarks. Though these KLT relations are more complicated than the pure gluon case, as stated in [26], it is possible to study them using similar idea as given in this paper.

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A. The generalized $U(1)$ -decoupling identity

In the proof of KK-relation, there is one new identity we need to use to show the vanishing of boundary

contributions. It is given by

$$\sum_{P(O(\alpha) \cup O(\beta))} A(1, P(O(\alpha) \cup O(\beta))) = 0 \quad (\text{A.1})$$

where the sum is over all permutations where relative ordering of two subsets α, β is kept. This identity has been shown to be the sum of two KK-relations, so it is true if the KK-relation is true. However, relation (A.1) has a physical picture as the generalized $U(1)$ -decoupling identity as we will show in this section. For simplicity we will focus on the pure gauge theory with $U(N)$ gauge group. The generator can be chosen as $N \times N$ matrix E_{ij} with number 1 at position (i, j) and zero at other places. These N^2 generators can be divided into following three categories: category (A) with $1 \leq i, j \leq N_1$; category (B) with $N_1 + 1 \leq i, j \leq N$ and category (C) with $1 \leq i \leq N_1, N_1 + 1 \leq j \leq N$ or $1 \leq j \leq N_1, N_1 + 1 \leq i \leq N$. The Lie bracket is $[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{il} E_{kj}$, so we have

$$[A, A] \sim A, \quad [B, B] \sim B, \quad [C, C] \sim A + B, \quad [A, B] \sim 0, \quad [A, C] \sim C, \quad [B, C] \sim C. \quad (\text{A.2})$$

The trace is given by $Tr(E_{ij} E_{kl}) = \delta_{jk} \delta_{il}$ so we have

$$tr(AA) \neq 0, \quad tr(BB) \neq 0, \quad tr(CC) \neq 0, \quad tr(AB) = 0, \quad tr(AC) = 0, \quad tr(BC) = 0. \quad (\text{A.3})$$

The trace structure (A.3) tells us that the propagator can only be the AA, BB, CC three types and there is no mixing between different categories. This observation is very important for our late arguments. Having (A.2) and (A.3), from the Lagrangian

$$Tr((\partial A + [A, A])(\partial A + [A, A])) \quad (\text{A.4})$$

we see that nonzero cubic vertex will be following types

$$AAA, BBB, ACC, BCC, \quad (\text{A.5})$$

and the nonzero four-point vertex will be following types

$$AAAA, BBBB, CCCC, AACC, BBCC, ABCC. \quad (\text{A.6})$$

Now we consider the amplitude with 1 belonging to type (C), set α belonging to type (A) and set β belonging to type (B). Under the color decomposition, the contribution to this particular color configuration is nothing, but the combination at the left hand side of (A.1). To show the amplitude is zero for above color configuration, let us study Feynman diagrams of tree-level amplitudes. Since the coexistence of type (A) and (B), there must be propagators of CC type, thus we see that each diagram must have a bone structure given by propagators of type CC . Since each nonzero vertex with (C)-type must have at least two legs with type (C) from (A.5) and (A.6) and the bone structure given by propagators of type CC must have at least two endpoints, there is no diagram for color configuration with only one external particle belonging to type (C). Thus the amplitude of this particular color configuration must be zero. If there are at least two external particles belonging to type (C), there are nonzero contributions from Feynman diagrams and we can not claim anything. Thus we have given a physical picture of (A.1) by our generalized $U(1)$ -decoupling argument.

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